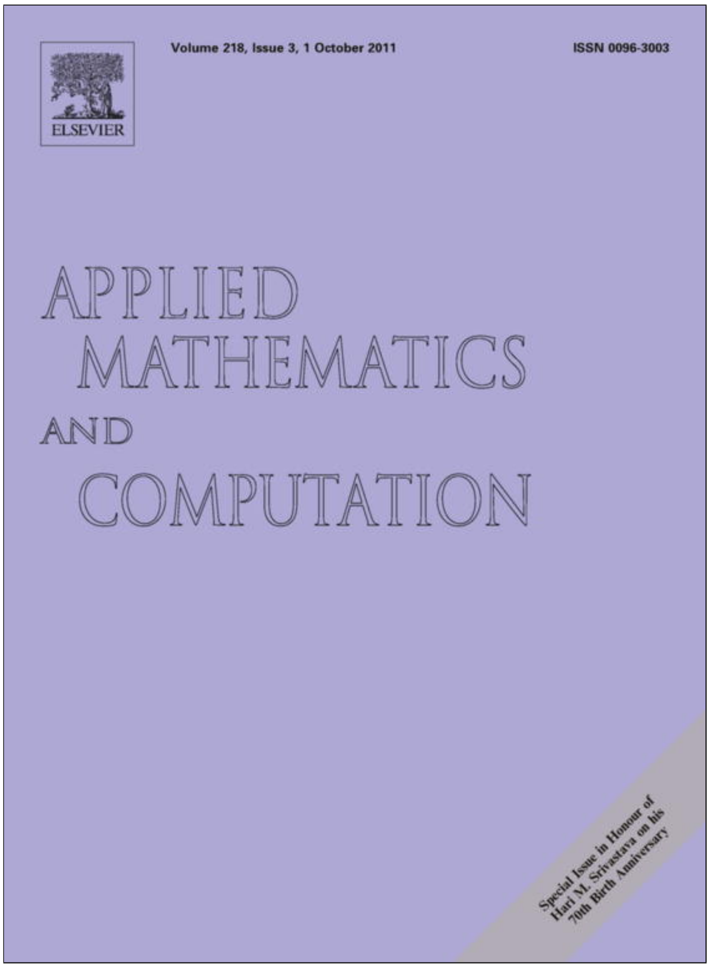
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Closure properties of operators on the Ma–Minda type starlike and convex functions ☆

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A R T I C L E   I N F O

Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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A B S T R A C T

A normalized univalent function f is called Ma–Minda starlike or convex if $zf'(z)/f(z) \prec \varphi(z)$ or $1+zf''(z)/f'(z) \prec \varphi(z)$ where φ is a convex univalent function with $\varphi(0)=1$. The class of Ma–Minda convex functions is shown to be closed under certain operators that are generalizations of previously studied operators. Analogous inclusion results are also obtained for subclasses of starlike and close-to-convex functions. Connections with various earlier works are made.

1. Introduction and motivation

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane and let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{D}$ normalized by the conditions $f(0)=0$ and $f'(0)=1$. An analytic function f is *subordinate* to g in $\mathbb{D}$, written $f(z) \prec g(z) \quad (z \in \mathbb{D})$, if there exists a function w analytic in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ satisfying $f(z)=g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is a starlike domain with respect to 0, and $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is a convex domain. Analytically, these geometric properties are respectively equivalent to the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{or } \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0.$$

In terms of subordination, these conditions are respectively equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad \text{or } 1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

The subclasses of $\mathcal{A}$ consisting of starlike and convex functions are denoted respectively by $\mathcal{ST}$ and $\mathcal{CV}$.

Ma and Minda [18] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $(1+z)/(1-z)$ by a more general analytic function φ with positive real part and normalized by the conditions $\varphi(0)=1$ and $\varphi'(0)>0$. Further it is assumed that φ maps the unit disk $\mathbb{D}$ onto a region starlike with respect

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to 1 that is symmetric with respect to the real axis. They introduced the following general classes that envelopes several well-known classes as special cases:

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

and

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

For $\beta < 1$, let $\varphi_\beta : \mathbb{D} \to \mathbb{C}$ be defined by

$$\varphi_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z}.$$

Then the classes $\mathcal{ST}(\varphi_\beta)$ and $\mathcal{CV}(\varphi_\beta)$ reduce to the familiar classes of univalent starlike and convex functions of order β:

$$\mathcal{ST}(\beta) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \beta \right\},$$
$$\mathcal{CV}(\beta) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta \right\}.$$

If $\varphi_{\lambda,\mu} : \mathbb{D} \to \mathbb{C}$ is the conformal mapping of $\mathbb{D}$ onto the domain

$$\Omega_{\lambda,\mu} = \{ w \in \mathbb{C} : \mathrm{Re} w - \mu \ > \ \lambda |w - 1| \},$$

and normalized by $\varphi_{\lambda,\mu}(0) = 1$, then the classes $\mathcal{ST}(\varphi_{\lambda,\mu})$ and $\mathcal{CV}(\varphi_{\lambda,\mu})$ reduce to the classes $\mathcal{ST}(\lambda,\mu)$ and $\mathcal{CV}(\lambda,\mu)$ of parabolic starlike and uniformly convex functions. The class $\mathcal{CV}(\lambda,\mu)$ was investigated by Yang and Owa [28], and Frasin [15].

In [3], Biernacki falsely claimed that $\int_0^z f(\zeta)/\zeta d\zeta$ is univalent whenever f is univalent. Moved by this, Causey [13] considered a related problem of finding conditions on δ such that $\int_0^z (f(\zeta)/\zeta)^\delta d\zeta$ is univalent whenever f is univalent. A survey on these problems can be found in [20]. In recent years, considerable attention has been given to the problem for various classes of univalent functions, see for example, the works of [1,2,4–12,15–17,22,21,26,27].

For $0 \leqslant \alpha \leqslant 1$, let $\mathcal{CC}_\alpha$ be the class of functions f analytic in $|z| < 1$ satisfying $|\arg(f'(z)/h'(z))| \leqslant \frac{1}{2}\alpha\pi$ in $\mathbb{D}$ with respect to some univalent convex function h (depending on f). Suppose $|z_1| \leqslant 1, |z_2| \leqslant 1$, and $f \in \mathcal{CC}_\alpha$. Then Pommerenke [23] proved that

$$\int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta} d\zeta \in \mathcal{CC}_\alpha.$$

Singh [25] showed that $\frac{1}{2}\int_0^z [f(t) - f(-t)]t^{-1} dt$ is starlike if f is starlike. Analogous results were also proven for convexity and close-to-convexity. Extending the results of Singh [25], Chandra and Singh [14] proved that the integral

$$\int_0^z \frac{f(e^{i\mu}\zeta) - f(e^{i\psi}\zeta)}{(e^{i\mu} - e^{i\psi})\zeta} d\zeta, \quad (\mu \neq \psi, 0 \leqslant \mu, \psi < 2\pi)$$

preserves membership in the classes of starlike, convex and close-to-convex functions. The integral operators discussed earlier are generalized in the following form:

Definition 1. For $\alpha_i \geqslant 0$ and $f_i \in \mathcal{A}$, define the operators

$$F(z) = F_{f_1,\cdots,f_n;z_1,z_2}(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(z_2\zeta) - f_i(z_1\zeta)}{(z_2 - z_1)\zeta} \right)^{\alpha_i} d\zeta \quad (z_1, z_2 \in \overline{\mathbb{D}}), \tag{1.1}$$

$$G(z) = G_{f_1,\cdots,f_n;z_1,z_2}(z) = z\prod_{i=1}^n \left(\frac{f_i(z_2 z) - f_i(z_1 z)}{(z_2 - z_1)z} \right)^{\alpha_i} \quad (z_1, z_2 \in \overline{\mathbb{D}}). \tag{1.2}$$

Here the powers are chosen to be principal. It is clear that $G(z) = zF'(z)$. With F and G as above, define the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ respectively by

$$\mathcal{F}_n(f_1, \ldots, f_n) := \{ F_{f_1,\cdots,f_n;z_1,z_2} : f_i \in \mathcal{A}, z_1, z_2 \in \overline{\mathbb{D}} \}, \tag{1.3}$$

and

$$\mathcal{G}_n(f_1, \ldots, f_n) := \{ G_{f_1,\cdots,f_n;z_1,z_2} : f_i \in \mathcal{A}, z_1, z_2 \in \overline{\mathbb{D}} \}. \tag{1.4}$$

In the case $n = 1$, it is assumed that $\alpha_1 = 1$ in (1.1) and (1.2), and we write $\mathcal{F}(f) := \mathcal{F}_1(f)$ and $\mathcal{G}(f) := \mathcal{G}_1(f)$ respectively.

Ponnusamy and Singh [24] introduced the operator F in (1.1) and investigated its univalence. In this paper, membership preservation properties of the operators F and G on the subclasses of starlike, convex and close-to-convex functions will be investigated. We shall also make connections with various earlier works. The following lemma will be required.

Lemma 1.1 [19]. *Let G and H be analytic in $\mathbb{D}$, and let H be univalent and convex. If the range of the function G'/H' is contained in a convex set Δ, then so do the numbers $[G(z_2)-G(z_1)]/[H(z_2)-H(z_1)]$ for $z_1, z_2 \in \mathbb{D}$.*

This lemma yields the following result:

Lemma 1.2. *Let φ be a convex function with $\varphi(0)=1$ and $z_1, z_2 \in \overline{\mathbb{D}}$. If $f \in \mathcal{A}$ satisfies the subordination $f'(z)/g'(z) \prec \varphi(z)$ for some $g \in \mathcal{CV}$, then*

$$\frac{f(z_2z)-f(z_1z)}{g(z_2z)-g(z_1z)} \prec \varphi(z).$$

2. The Operators *F* and *G* on subclasses of convex functions

Theorem 2.1. *For $i=1,2,\ldots,n$, let $\alpha_i \geqslant 0,\ 0 \leqslant \beta_i < 1$ and $\gamma := 1-\sum_{i=1}^n \alpha_i(1-\beta_i)$. For $f_i \in \mathcal{A}$, let F and G be given by (1.1) and (1.2) respectively. If $f_i \in \mathcal{CV}(\beta_i)$, then $F \in \mathcal{CV}(\gamma)$ and $G \in \mathcal{ST}(\gamma)$. In particular, if $\sum_{i=1}^n \alpha_i(1-\beta_i) \leqslant 1$, then $F \in \mathcal{CV}$ and $G \in \mathcal{ST}$.*

Proof. Let $f_i \in \mathcal{CV}(\beta_i)$ so that

$$\frac{(zf_i'(z))'}{f_i'(z)} = 1 + \frac{zf_i''(z)}{f_i'(z)} \prec \varphi_{\beta_i}(z), \tag{2.1}$$

where $\varphi_{\beta_i} : \mathbb{D} \to \mathbb{C}$ is the convex function defined by

$$\varphi_{\beta_i}(z) = \frac{1+(1-2\beta_i)z}{1-z}.$$

For $0 \leqslant \beta_i < 1,\ \varphi_{\beta_i}(\mathbb{D})$ is the half-plane $\mathrm{Re}\, w > \beta_i$ and hence $\varphi_{\beta_i}(\mathbb{D})$ is a convex domain. Since f_i is a convex function, Lemma 1.2 applied to the subordination (2.1) yields

$$\frac{z_2zf_i'(z_2z)-z_1zf_i'(z_1z)}{f_i(z_2z)-f_i(z_1z)} \prec \varphi_{\beta_i}(z),$$

or equivalently,

$$\mathrm{Re}\left\{\frac{z_2zf_i'(z_2z)-z_1zf_i'(z_1z)}{f_i(z_2z)-f_i(z_1z)}\right\} > \beta_i. \tag{2.2}$$

A differentiation of (1.1) yields

$$F'(z) = \prod_{i=1}^n \left(\frac{f_i(z_2z)-f_i(z_1z)}{(z_2-z_1)z}\right)^{\alpha_i},$$

and differentiating logarithmically shows that

$$1+\frac{zF''(z)}{F'(z)} = \left(1-\sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \left(\frac{z_2zf_i'(z_2z)-z_1zf_i'(z_1z)}{f_i(z_2z)-f_i(z_1z)}\right). \tag{2.3}$$

It follows from (2.3) by using the inequality (2.2) that $F \in \mathcal{CV}(\gamma)$:

$$1+\mathrm{Re}\frac{zF''(z)}{F'(z)} = \left(1-\sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \mathrm{Re}\left(\frac{z_2zf_i'(z_2z)-z_1zf_i'(z_1z)}{f_i(z_2z)-f_i(z_1z)}\right) > \left(1-\sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i\beta_i = \gamma.$$

The result that $G \in \mathcal{ST}(\gamma)$ follows from the fact that $zF'(z) = G(z)$ and that $F \in \mathcal{CV}(\gamma)$. $\square$

Corollary 2.1. *Let $0 \leqslant \beta < 1$. For $i=1,2,\ldots,n$, let $\alpha_i \geqslant 0$ and $\gamma := 1-(1-\beta)\sum_{i=1}^n \alpha_i$. For $f_i \in \mathcal{A}$, let F be given by (1.1). If $f_i \in \mathcal{CV}(\beta)$, then $F \in \mathcal{CV}(\gamma)$ and $G \in \mathcal{ST}(\gamma)$. In particular, if $\sum_{i=1}^n \alpha_i \leqslant 1$, then $F \in \mathcal{CV}(\beta)$ and $G \in \mathcal{ST}(\beta)$.*

Corollary 2.2. For $i=1,2,\ldots,n$, let $\alpha_i \geqslant 0,\ 0 \leqslant \beta_i < 1$ and $\gamma := 1-\sum_{i=1}^n \alpha_i(1-\beta_i)$. For $f_i \in \mathcal{A}$, let $\mathcal{F}_n(f_1,\ldots,f_n)$ and $\mathcal{G}_n(f_1,\ldots,f_n)$ be given by (1.3) and (1.4) respectively. If $f_i \in \mathcal{CV}(\beta_i)$, then $\mathcal{F}_n(f_1,\ldots,f_n) \subset \mathcal{CV}(\gamma)$ and $\mathcal{G}_n(f_1,\ldots,f_n) \subset \mathcal{ST}(\gamma)$. In particular, if $\sum_{i=1}^n \alpha_i(1-\beta_i) \leqslant 1$, then $\mathcal{F}_n(f_1,\ldots,f_n) \subset \mathcal{CV}$ and $\mathcal{G}_n(f_1,\ldots,f_n) \subset \mathcal{ST}$. Also if $f \in \mathcal{CV}(\alpha),\ 0 \leqslant \alpha < 1$, then $\mathcal{F}(f) \subset \mathcal{CV}(\alpha)$ and $\mathcal{G}(f) \subset \mathcal{ST}(\alpha)$.

Remark 2.1. Let $0 \leqslant \alpha < 1$. The inclusions $\mathcal{G}(f) \subset \mathcal{ST}(\alpha)$ and $\mathcal{F}(f) \subset \mathcal{CV}(\alpha)$ for $f \in \mathcal{CV}(\alpha)$ contained the results of Chandra and Singh [14, Theorem 2.1, p. 1271 and Theorem 2.4 p. 1273]. The following result of Singh [25] is also contained in ours: If $f \in \mathcal{CV}$, then $\int_0^z (f(t) - f(-t))/(2t)dt \in \mathcal{CV}$.

For $i = 1, 2, \ldots, n$, let $\alpha_i \geqslant 0, 0 \leqslant \beta < 1$ and $\sum_{i=1}^n \alpha_i = 1$. For $f_i \in \mathcal{A}$, let F be given by (1.1). By Corollary 2.1, if $f_i \in \mathcal{CV}(\beta)$, then $F \in \mathcal{CV}(\beta)$. This result is next proved in a more general setting:

Theorem 2.2. *For $i = 1, 2, \ldots, n$, let $\alpha_i \geqslant 0$ and $\sum_{i=1}^n \alpha_i \leqslant 1$. Let φ be convex in $\mathbb{D}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f_i \in \mathcal{CV}(\varphi)$, then F given by (1.1) satisfies $F \in \mathcal{CV}(\varphi)$, and G given by (1.2) satisfies $G \in \mathcal{ST}(\varphi)$.*

Proof. Let $f_i \in \mathcal{CV}(\varphi)$ so that

$$1 + \frac{zf_i''(z)}{f_i'(z)} \prec \varphi(z).$$

Since φ is a function with positive real part, it follows that

$$1 + \mathrm{Re}\frac{zf_i''(z)}{f_i'(z)} > 0,$$

and hence f_i is a convex function. As shown in the proof of Theorem 2.1, Lemma 1.2 yields

$$\frac{z_2 z f_i'(z_2 z) - z_1 z f_i'(z_1 z)}{f_i(z_2 z) - f_i(z_1 z)} \prec \varphi(z),$$

or for any fixed $z \in \mathbb{D}$,

$$\frac{z_2 z f_i'(z_2 z) - z_1 z f_i'(z_1 z)}{f_i(z_2 z) - f_i(z_1 z)} \in \varphi(\mathbb{D}).$$

Since φ is convex, and $1 = \varphi(0) \in \varphi(\mathbb{D})$, the convex combination of $n+1$ complex numbers

$$1; \frac{z_2 z f_i'(z_2 z) - z_1 z f_i'(z_1 z)}{f_i(z_2 z) - f_i(z_1 z)} \quad (i = 1, 2, \ldots, n),$$

is again in $\varphi(\mathbb{D})$:

$$(1 - \sum_{i=1}^n \alpha_i)(1) + \sum_{i=1}^n \alpha_i \left(\frac{z_2 z f_i'(z_2 z) - z_1 z f_i'(z_1 z)}{f_i(z_2 z) - f_i(z_1 z)} \right) \in \varphi(\mathbb{D}).$$

Thus it follows that

$$\left(1 - \sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \left(\frac{z_2 z f_i'(z_2 z) - z_1 z f_i'(z_1 z)}{f_i(z_2 z) - f_i(z_1 z)} \right) \prec \varphi(z).$$

In view of (2.3), the above subordination becomes

$$1 + \frac{zF''(z)}{F'(z)} \prec \varphi(z),$$

which proves $F \in \mathcal{CV}(\varphi)$. $\square$

Corollary 2.3. *Let φ be convex in $\mathbb{D}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f \in \mathcal{CV}(\varphi)$, then*

$$\int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta} d\zeta \in \mathcal{CV}(\varphi) \quad \text{and} \quad \frac{f(z_2 z) - f(z_1 z)}{z_2 - z_1} \in \mathcal{ST}(\varphi).$$

Corollary 2.4. *For $i = 1, 2, \ldots, n$, let $\alpha_i \geqslant 0$ and $\sum_{i=1}^n \alpha_i \leqslant 1$. Let φ be convex in $\mathbb{D}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f_i \in \mathcal{CV}(\varphi)$, then $\mathcal{F}_n(f_1, \ldots, f_n) \subset \mathcal{CV}(\varphi)$ and $\mathcal{G}_n(f_1, \ldots, f_n) \subset \mathcal{ST}(\varphi)$. In particular, if $f \in \mathcal{CV}(\varphi)$, then $\mathcal{F}(f) \subset \mathcal{CV}(\varphi)$ and $\mathcal{G}(f) \subset \mathcal{ST}(\varphi)$.*

3. Operators on subclasses of starlike and close-to-convex functions

In this section, we shall devote attention to the following special case of the operator F:

$$F_1(z) := \int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta} d\zeta.$$

Theorem 3.1. *Let φ be convex in $\mathbb{D}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f \in \mathcal{ST}(\varphi)$, then $F_1 \in \mathcal{ST}(\varphi)$.*

Proof. Since $f \in \mathcal{ST}(\varphi)$, there exists a function $g \in \mathcal{CV}(\varphi)$ such that $f(z) = zg'(z)$. In fact, such a function g satisfies

$$g(\alpha z) = \int_0^z \frac{f(\alpha\zeta)}{\zeta} d\zeta \quad (|\alpha| \leqslant 1).$$

Using this identity, it follows that

$$F_1(z) = \int_0^z \frac{f(z_2\zeta) - f(z_1\zeta)}{(z_2 - z_1)\zeta} d\zeta = \frac{g(z_2 z) - g(z_1 z)}{z_2 - z_1}.$$

Since $g \in \mathcal{CV}(\varphi)$, Corollary 2.3 shows that

$$\frac{g(z_2 z) - g(z_1 z)}{z_2 - z_1} \in \mathcal{ST}(\varphi),$$

and hence $F_1 \in \mathcal{ST}(\varphi)$. □

Corollary 3.1. *Let φ be convex in $\mathbb{D}$ with positive real part, and normalized by $\varphi(0) = 1$. If $f \in \mathcal{ST}(\varphi)$, then $\mathcal{F}(f) \subset \mathcal{ST}(\varphi)$.*

Remark 3.1. For $0 \leqslant \alpha < 1$, if $f \in \mathcal{ST}(\alpha)$, then the above corollary shows that $\mathcal{F}(f) \subset \mathcal{ST}(\alpha)$. This result contains a result of [14, Theorem 2.3, p. 1273]. The above corollary also contains the following result of [25]: If $f \in \mathcal{ST}$, then $F(f) = \int_0^z (f(t) - f(-t))/(2t) dt \in \mathcal{ST}$.

Definition 2. Let φ and ψ be convex functions with positive real part and normalized respectively by $\phi(0) = 1$ and $\psi(0) = 1$. The class $\mathcal{CC}(\varphi, \psi)$ consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{f'(z)}{h'(z)} \prec \varphi(z),$$

where $h \in \mathcal{CV}(\psi)$.

For $0 \leqslant \alpha,\ \tau < 1$, let $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$ and $\psi_\tau : \mathbb{D} \to \mathbb{C}$ be defined by

$$\varphi_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad \psi_\alpha(z) = \frac{1 + (1 - 2\tau)z}{1 - z}.$$

In this case, the class $\mathcal{CC}(\varphi, \psi)$ reduces to the familiar class of univalent close-to-convex functions of order α and type τ:

$$\mathcal{CC}(\alpha, \tau) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{f'(z)}{h'(z)}\right) > \alpha, \text{where } h \in \mathcal{CV}(\tau) \right\}.$$

In this form, the class $\mathcal{CC}_\alpha$ investigated by Pommerenke [23] becomes a special case of $\mathcal{CC}(\varphi, \psi)$, that is,

$$\mathcal{CC}_\alpha = \mathcal{CC}\left(\left(\frac{1+z}{1-z}\right)^\alpha, \frac{1+z}{1-z}\right).$$

The following closure property for the class $\mathcal{CC}(\varphi, \psi)$ contains a result of Pommerenke [23].

Theorem 3.2. *If $f \in \mathcal{CC}(\varphi, \psi)$, then $F_1 \in \mathcal{CC}(\varphi, \psi)$.*

Proof. If $f \in \mathcal{CC}(\varphi, \psi)$, then there exists a function $h \in \mathcal{CV}(\psi)$ such that

$$\frac{f'(z)}{h'(z)} \prec \varphi(z).$$

Corollary 2.3 yields

$$H_1(z) := \int_0^z \frac{h(z_2\zeta) - h(z_1\zeta)}{(z_2 - z_1)\zeta} d\zeta \in \mathcal{CV}(\psi).$$

Since $\mathrm{Re}\psi(z) > 0$, the function h is convex. It follows from Lemma 1.2 that

$$\frac{f(z_2 z) - f(z_1 z)}{h(z_2 z) - h(z_1 z)} \prec \varphi(z).$$

Since

$$\frac{F_1'(z)}{H_1'(z)} = \frac{f(z_2 z) - f(z_1 z)}{h(z_2 z) - h(z_1 z)},$$

we deduce that $F_1 \in \mathcal{CC}(\varphi, \psi)$. $\square$

Corollary 3.2. *If $f \in \mathcal{CC}(\varphi, \psi)$, then $\mathcal{F}(f) \subset \mathcal{CC}(\varphi, \psi)$. In particular, for $0 \leqslant \alpha,\ \tau < 1$, if $f \in \mathcal{CC}(\alpha, \tau)$, then $\mathcal{F}(f) \subset \mathcal{CC}(\alpha, \tau)$.*

Remark 3.2. The second statement of the above corollary contains a result of [14, Theorem 2.6, p. 1274]. The above corollary also contains the following result of [25]: If $f \in \mathcal{CC}$ with respect to the convex function h, then $F(f) = \int_0^z (f(t) - f(-t))/(2t)dt \in \mathcal{CC}$ with respect to the convex function $H(f) = \int_0^z (h(t) - h(-t))/(2t)dt$.

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